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Non-Bayesian and Bayesian Prediction for Additive Flexible Weibull Extension-Lomax Distribution

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ABSTRACT
Prediction of future observations is an important problem in many practical applications. This paper focuses on considering one-sample and two-sample prediction (as a special case of the multi-sample prediction) for a future observation from additive flexible Weibull extension-Lomax distribution. Non-Bayesian and Bayesian prediction based on Type II censoring scheme are studied. The conditional prediction approach is discussed as a non-Bayesian prediction method. Also, Bayesian prediction is obtained under two different loss functions, the squared error and linear-exponential loss functions. Moreover, a simulation study is conducted to evaluate the performance of the derived predictors and three applications of COVID -19 data in some countries are considered.

KEYWORDS
Additive flexible Weibull extension-Lomax distribution, one-sample prediction, two-sample prediction, conditional prediction approach, Bayesian prediction, squared error loss function, linear-exponential loss function.

1. Introduction

Prediction of future observations is an important problem in many practical applications such as agricultural, industrial, demographic, medical, biological and engineering experiments. Many researchers have considered prediction for future observation from different lifetime distributions. Singh *et al.* (2013) studied one-sample and two-sample Bayesian prediction for future observation from Type-II censored sample from *flexible Weibull extension* (FWE) distribution. Valiollahi *et al.* (2017) considered one-sample prediction of future observation based on Type I and Type II hybrid censored samples from a two-parameter generalized exponential distribution. They used the likelihood and conditional prediction approaches as non-Bayesian and Bayesian prediction methods under the *squared error* (SE) loss function.

Sen *et al.* (2018) investigated the problem of a one-sample prediction of a future observation from the lognormal distribution using Bayesian and non-Bayesian methods based on Type I progressive hybrid censoring. Furthermore, Valiollahi *et al.* (2019) studied the one-sample prediction of a future observation based on Type I hybrid censored sample using the likelihood and the conditional prediction approach. Recently,

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Ateya *et al.* (2022) studied the one-sample and two-sample prediction schemes for a future observation from Burr X distribution based on a unified hybrid censoring scheme using the likelihood and the Bayesian prediction methods.

There are different types of lifetime data in reliability studies, lifetime testing, human mortality studies, engineering modeling, electronic sciences and biological surveys. Hence, different shapes of lifetime distributions are required for fitting these types of lifetime data. Researchers have proposed several extensions and modifications to provide more flexibility than the existing distributions. Therefore, a lot of references presented several methods for constructing, extending and generalizing lifetime distributions such as: the transformations of variables and distribution functions, probability integral transforms, compound distributions, finite and infinite mixture distributions and competing risks approach. [see Lai (2013)].

Competing risks often arise when there are more than one cause or mode of failure. These failure modes in some sense compete to cause the failure of the experimental unit, which is known in statistical literature as competing risks. Also, competing risks can occur frequently in series systems, in which their components are arranged in series. The lifetime of each component has a certain distribution with certain parameters. Assuming that the lifetimes of the components of the series system are statistically independent, then the lifetime of the series system can be obtained as the minimum of its components lifetimes. Furthermore, there are many lifetime distributions that were constructed based on the competing risks approach.

Some references in the field of the competing risks include Xie and Lai (1995), Wang (2000), Almalki and Yuan (2013), He *et al.* (2016), Oluyede *et al.* (2016), Singh (2016), Mdlongwa *et al.* (2017), Tarvirdizade and Ahmedpour (2019), Shakhatreh *et al.* (2019), Osagie and Osemwenkhae (2020), Kamal and Ismail (2020), Thach and Bris (2021), Makubate *et al.* (2021), Abba *et al.* (2022), Xavier *et al.* (2022) and Thach (2022).

Recently, Salem *et al.* (2022) introduced the *additive flexible Weibull extension-Lomax* (AFWE-L) distribution by considering a series system with two components functioning independently in series. The lifetime of the first component, X_1, has the FWE distribution with parameters α and β, the FWE distribution was obtained by Bebbington *et al.* (2007), and the lifetime of the second component, X_2, has the *Lomax* (L) distribution with parameters λ and θ. Therefore, the lifetime of the system is $X=\min\{X_1,\, X_2\}$ has AFWE-L distribution with parameter vector $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$. The *probability density function* (pdf), *cumulative distribution function* (cdf), *reliability function* (rf) and *hazard rate function* (hrf) of AFWE-L distribution are given, respectively, by:

$$f\left(x;\underline{\psi}\right)=\left[\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}+\frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}\right]e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta},\ x>0;\underline{\psi}>\underline{0}, \tag{1}$$

$$F\left(x;\underline{\psi}\right)=1-e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta}, \qquad x>0;\ \underline{\psi}>\underline{0}, \tag{2}$$

$$R\left(x;\underline{\psi}\right)=e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta}, \qquad x>0;\ \underline{\psi}>\underline{0}, \tag{3}$$

and

$$h\left(x;\underline{\psi}\right)=\left(\alpha+\frac{\beta}{x^{2}}\right)e^{\alpha x-\frac{\beta}{x}}+\frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}, \qquad x>0;\ \underline{\psi}>\underline{0}, \tag{4}$$

AFWE-L distribution has highly flexibility and diversity in the shapes of the pdf as well as the hrf which allow this distribution to fit several types of lifetime data. Since, its pdf exhibit decreasing, unimodal and decreasing-unimodal shapes and its hrf distribution can be increasing, decreasing, bathtub, bi-bathtub and modified bathtub shapes. Salem *et al.* (2022) derived some main properties of AFWE-L distribution and estimated the model parameters, rf and hrf using the *maximum likelihood* (ML) method based on Type II censoring scheme. Also, they obtained the asymptotic confidence intervals of the parameters, rf and hrf. Moreover, they demonstrated the superiority of AFWE-L distribution over some existing distributions through three applications to COVID-19 data in some countries. Bayesian estimation of the AFWE-L distribution parameters, rf, hrf and *reversed hazard rate function* (rhrf) is discussed by Abd EL-Kader *et al.* (2022) under the SE loss function as a symmetric loss function and the *linear-exponential* (LINEX) loss function as an asymmetric loss function based on Type II censoring scheme.

The rest of this paper is organized as follows: in Section 2, different one-sample predictors for future observation from AFWE-L distribution based on Type II censoring scheme are obtained using non-Bayesian and Bayesian approaches. Also, confidence and credible intervals are discussed. Non-Bayes and Bayes two-sample predictors (point and interval) for future observation from AFWE-L distribution based on Type II censoring scheme are derived in Section 3. In Section 4, a simulation study is conducted to evaluate the performance of the derived predictors. Finally, three applications to COVID-19 data in some countries are given in Section 5.

2. One-Sample Prediction

In this section, non-Bayes and Bayes one-sample predictors for future observation from AFWE-L distribution based on Type II censoring scheme are obtained.

Let $X_{(1)}, X_{(2)}, \cdots, X_{(r)}; r \leq n$ represent the observed ordered lifetimes of a censored Type II sample (the informative sample) from AFWE-L distribution and $X_{(r+1)}, X_{(r+2)}, \cdots, X_{(n)}$ is the unobserved future ordered lifetimes (the future sample) from the same distribution. This section aims is to derive predictors of the unobserved future ordered statistic $Y_{(s)} = X_{(s)}$ for $r < s \leq n$.

The conditional pdf of $Y_{(s)}$ given $\underline{x} = x_{(1)}, \ . \ . \ . \ , \ x_{(r)}$ is

$$f\left(y_{(s)} \mid \underline{x};\underline{\psi}\right)= C_{r,s,n} f\left(y_{(s)};\underline{\psi}\right)\left[R\left(x_{(r)};\underline{\psi}\right)-R\left(y_{(s)};\underline{\psi}\right)\right]^{s-r-1}\frac{\left[R\left(y_{(s)};\underline{\psi}\right)\right]^{n-s}}{\left[R\left(x_{(r)};\underline{\psi}\right)\right]^{n-r}}, \quad y_{(s)}>x_{(r)}, \tag{5}$$

where

$$C_{r,s,n}=\frac{(n-r)!}{(s-r-1)!\,(n-s)!}.$$

Since,

$$f\left(y_{(s)};\underline{\psi}\right)=h\left(y_{(s)};\underline{\psi}\right)R\left(y_{(s)};\underline{\psi}\right),$$

and by using the binomial expansion of $\left[R\left(x_r;\underline{\psi}\right)-R\left(y_{(s)};\underline{\psi}\right)\right]^{s-r-1}$ in (5), one obtains

$$\left[R\left(x_{(r)};\underline{\psi}\right)-R\left(y_{(s)};\underline{\psi}\right)\right]^{s-r-1}=\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^j\left[R\left(y_{(s)};\underline{\psi}\right)\right]^j\left[R\left(x_{(r)};\underline{\psi}\right)\right]^{s-r-j-1}.$$

Hence, (5) can be rewritten as:

$$f\left(y_{(s)}\mid\underline{x};\underline{\psi}\right)=\sum_{j=0}^{s-r-1}C_{r,s,n,j}h\left(y_{(s)};\underline{\psi}\right)\left[\frac{R\left(y_{(s)};\underline{\psi}\right)}{R\left(x_{(r)};\underline{\psi}\right)}\right]^{n-s+j+1},\qquad y_{(s)}>x_{(r)},\quad(6)$$

where

$$C_{r,s,n,j}=\frac{(n-r)!(-1)^j}{j!\,(s-r-j-1)!\,(n-s)!}.\qquad(7)$$

Substituting (3) and (4) into (6), then the conditional pdf of $Y_{(s)}$ given $\underline{x}=x_{(1)},\ldots,\ x_{(r)}$ is given by:

$$f\left(y_{(s)}\mid\underline{x};\underline{\psi}\right)=\sum_{j=0}^{s-r-1}C_{r,s,n,j}h\left(y_{(s)};\underline{\psi}\right)\exp\left\{-(n-s+j+1)\left[e^{\alpha y_{(s)}-\frac{\beta}{y_{(s)}}}-u_{(r)}+\theta\ln\left(1+\frac{y_{(s)}}{\lambda}\right)-\theta\ln w_{(r)}\right]\right\},\qquad y_{(s)}>x_{(r)},\quad(8)$$

where $C_{r,s,n,j}$ is given by (7),

$$u_{(r)}=e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}},\qquad(9)$$

$$w_{(r)}=\left(1+\frac{x_{(r)}}{\lambda}\right),\qquad(10)$$

and

$$h\left(y_s;\underline{\psi}\right)=\left[\left(\alpha+\frac{\beta}{y_{(s)}^2}\right)e^{\alpha y_{(s)}-\frac{\beta}{y_{(s)}}}+\frac{\theta}{\lambda}\left(1+\frac{y_{(s)}}{\lambda}\right)^{-1}\right].\qquad(11)$$

2.1. Conditional prediction approach

In this subsection, one-sample conditional prediction approach is applied to derive point and interval predictors of $Y_{(s)}$.

a. Point prediction

The conditional predictor of $Y_{(s)}$; $\hat{y}_{(s)C}$, can be derived using (8), then using the invariance property of the ML estimators, the parameters $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$ will be replaced by their ML estimators, $\underline{\widehat{\psi}}=\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda},\widehat{\theta}\right)$, which were derived by Salem *et al.* (2022) as follows:

$$f\left(y_{(s)}\,\Big|\,\underline{x};\underline{\widehat{\psi}}\right)=\sum_{j=0}^{s-r-1}C_{r,s,n,j}h\left(y_{(s)};\underline{\widehat{\psi}}\right)\times\exp\left\{-(n-s+j+1)\left[e^{\widehat{\alpha}y_{(s)}-\frac{\widehat{\beta}}{y_{(s)}}}-\hat{u}_{(r)}+\widehat{\theta}\ln\left(1+\frac{y_{(s)}}{\widehat{\lambda}}\right)-\widehat{\theta}\ln\hat{w}_{(r)}\right]\right\},\quad y_{(s)}>x_{(r)},\tag{12}$$

where$C_{r,s,n,j}$ is given by (7),

$$\hat{u}_{(r)}=e^{\widehat{\alpha}x_{(r)}-\frac{\widehat{\beta}}{x_{(r)}}},$$

$$\hat{w}_{(r)}=\left(1+\frac{x_{(r)}}{\widehat{\lambda}}\right),$$

and

$$h\left(y_{(s)};\underline{\widehat{\psi}}\right)=\left[\left(\widehat{\alpha}+\frac{\widehat{\beta}}{y_{(s)}^2}\right)e^{\widehat{\alpha}y_{(s)}-\frac{\widehat{\beta}}{y_{(s)}}}+\frac{\widehat{\theta}}{\widehat{\lambda}}\left(1+\frac{y_{(s)}}{\widehat{\lambda}}\right)^{-1}\right].$$

Then, the conditional predictor $\hat{y}_{(s)}$ can be obtained as given below

$$\hat{y}_{(s)}=E\left(y_{(s)}\,\Big|\,\underline{\widehat{\psi}}\right)=\int_{y_{(s)}}y_{(s)}f\left(y_{(s)}\,\Big|\,\underline{x};\underline{\widehat{\psi}}\right)dy_{(s)}=\int_{x_{(r)}}^{\infty}y_{(s)}\sum_{j=0}^{s-r-1}C_{r,s,n,j}h\left(y_{(s)};\underline{\widehat{\psi}}\right)\times\exp\left\{-(n-s+j+1)\left[e^{\widehat{\alpha}y_{(s)}-\frac{\widehat{\beta}}{y_{(s)}}}-\hat{u}_{(r)}+\widehat{\theta}\ln\left(1+\frac{y_{(s)}}{\widehat{\lambda}}\right)-\widehat{\theta}\ln\hat{w}_{(r)}\right]\right\}dy_{(s)}.\tag{13}$$

Numerical method can be used to obtain $\hat{y}_{(s)}$.

b. Interval prediction

Using the conditional pdf of the s^{th} order statistic given $\underline{x}=x_{(1)},\ldots,x_{(r)}$ and by replacing the parameters $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$ by their ML estimators $\underline{\widehat{\psi}}=\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda},\widehat{\theta}\right)$ as in (12), the $(1-\omega)\,100\%$ conditional predictive interval for the future ordered failure, $Y_{(s)}$ can be derived from the following probability:

$$P\left[L_{C1}\left(\underline{x}\right) < Y_{(s)} < U_{C1}\left(\underline{x}\right) \left|\underline{x}\right.\right] = 1-\omega, \tag{14}$$

where $L_{C1}\left(\underline{x}\right)$ and $U_{C1}\left(\underline{x}\right)$ are the lower and upper bounds of the prediction interval based on the conditional prediction approach.
The conditional predictive bounds, $\left(L_{C1}\left(\underline{x}\right),\ U_{C1}\left(\underline{x}\right)\right)$, can be obtained as given below:

$$P\left[Y_{(s)} > L_{C1}\left(\underline{x}\right) \left|\underline{x}\right.\right] = \int_{L_{C1}(\underline{x})}^{\infty} f\left(y_{(s)} \,\middle|\, \underline{x};\widehat{\underline{\psi}}\right) dy_{(s)} = 1-\frac{\omega}{2}, \tag{15}$$

and

$$P\left[Y_{(s)} > U_{C1}\left(\underline{x}\right) \left|\underline{x}\right.\right] = \int_{U_{C1}(\underline{x})}^{\infty} f\left(y_{(s)} \,\middle|\, \underline{x};\widehat{\underline{\psi}}\right) dy_{(s)} = \frac{\omega}{2}. \tag{16}$$

Substituting (12) into (15) and (16), then the lower and upper bounds of the prediction interval of $Y_{(s)}$ can be calculated numerically.

2.2. Bayesian prediction

In this subsection, one-sample Bayesian prediction of the future ordered failure, $Y_{(s)}$, from AFWE-L distribution based on Type II censoring scheme is derived under the SE loss function as a symmetric loss function and under the LINEX loss function as an asymmetric loss function. Also, credible interval of $Y_{(s)}$ is obtained.

Suppose that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)}$ is a censored sample of size r from AFWE-L distribution with parameter vector $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$, then the likelihood function can be obtained as follows:

$$L\left(\underline{\psi} \mid \underline{x}\right) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)} \exp\left\{-\left(n-r\right)u_{(r)} - \sum_{i=1}^{r} u_{(i)}\right\}, \tag{17}$$

where

$$h\left(x_{(i)};\underline{\psi}\right) = \left[\left(\alpha + \frac{\beta}{x_{(i)}^{2}}\right) e^{\alpha x_{(i)} - \frac{\beta}{x_{(i)}}} + \frac{\theta}{\lambda}\left(1+\frac{x_{(i)}}{\lambda}\right)^{-1}\right]. \tag{18}$$

$$u_{(i)} = e^{\alpha x_{(i)} - \frac{\beta}{x_{(i)}}}, \tag{19}$$

$$w_{(i)} = \left(1+\frac{x_{(i)}}{\lambda}\right), \tag{20}$$

and $u_{(r)}$ and $w_{(r)}$ are given in (9) and (10), respectively.
Using the joint prior distribution suggested by Abd EL-Kader *et al.* (2022) as follows:

$$\pi\left(\underline{\psi}\right) \propto \alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}e^{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)}, \quad \alpha,\beta,\lambda,\theta>0; a_1,a_3>-1, a_2,a_4,b_1,b_2,b_3,b_4>0, \tag{21}$$

where a_i and b_i, $i = i, 2, 3, 4$, are the hyperparameters of the joint prior distribution.

This joint prior distribution was derived by assuming that the parameters$\underline{\psi}=(\alpha,\beta,\lambda,\theta) = (\psi_1,\psi_2,\psi_3,\psi_4)$ of the AFWE-L distribution are unknown random variables and the joint prior of the parameters α and β is independent of the joint prior of the parameters λ and θ. Then, the joint prior distribution of $\underline{\psi}$ is

$$\pi\left(\underline{\psi}\right) = \pi(\alpha,\beta)\,\pi(\lambda,\theta). \tag{22}$$

By assuming that the parameters α and β are dependent with a joint bivariate prior distribution that was used by AL-Hussaini and Jaheen (1992), which is expressed as:

$$\pi(\alpha,\beta) = \pi(\alpha \mid \beta)\,\pi(\beta), \tag{23}$$

where

$$\pi(\alpha \mid \beta) = \frac{\alpha^{a_1}\beta^{a_1+1}}{\Gamma(a_1+1)\; b_1^{a_1+1}}e^{-\frac{\alpha\beta}{b_1}}, \quad \alpha,\beta>0; a_1>-1, b_1>0, \tag{24}$$

and the marginal prior distribution of β is a gamma prior distribution with parameters $a_2$and b_2 and the following pdf:

$$\pi(\beta) = \frac{\beta^{a_2-1}}{\Gamma(a_2)\; {b_2}^{a_2}}e^{-\frac{\beta}{b_2}}, \qquad \beta>0; a_2,b_2>0. \tag{25}$$

Then, the joint prior distribution of α and β can be obtained by substituting (24) and (25) into (23) as follows:

$$\pi(\alpha,\beta) \propto \alpha^{a_1}\beta^{a_1+a_2}e^{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)}, \quad \alpha,\beta>0; a_1>-1, a_2,b_1,b_2>0. \tag{26}$$

Similarly, by assuming that λ and θ are dependent with the following joint bivariate prior distribution:

$$pi(\lambda,\theta) = \pi(\lambda \mid \theta)\,\pi(\theta), \tag{27}$$

where

$$\pi(\lambda \mid \theta) = \frac{\lambda^{a_3}\theta^{a_3+1}}{\Gamma(a_3+1)\; b_3^{a_3+1}}e^{-\frac{\lambda\theta}{b_3}}, \quad \lambda,\theta>0; a_3>-1, b_3>0, \tag{28}$$

and the marginal prior distribution of θ is a gamma prior distribution with parameters

$a_4$and b_4 and the following pdf:

$$\pi(\theta) = \frac{\theta^{a_4-1}}{\Gamma(a_4)\ {b_4}^{a_4}} e^{-\frac{\theta}{b_4}}, \qquad \theta > 0; a_4, b_4 > 0. \tag{29}$$

Hence, the joint prior distribution of λ and θ can be given by substituting (28) and (29) into (27) as:

$$\pi(\lambda, \theta) \propto \lambda^{a_3} \theta^{a_3+a_4} e^{-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)}, \quad \lambda, \theta > 0; a_3 > -1, a_4, b_3, b_4 > 0. \tag{30}$$

Substituting (26) and (30) into (22) the joint prior distribution of $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$ can be obtained which is given in (21).
The joint posterior distribution of $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$ can be formed using (17) and (21) as:

$$\pi\left(\underline{\psi} \mid \underline{x}\right) = A\ L\left(\underline{\psi} \mid \underline{x}\right) \pi\left(\underline{\psi}\right), \tag{31}$$

where
A is the normalizing constant defined by:

$$\mathrm{A}^{-1} = \int_{\underline{\psi}} L\left(\underline{\psi} \mid \underline{x}\right) \pi\left(\underline{\psi}\right) d\underline{\psi}, \tag{32}$$

where

$$\int_{\underline{\psi}} = \int_{\alpha=0}^{\infty} \int_{\beta=0}^{\infty} \int_{\lambda=0}^{\infty} \int_{\theta=0}^{\infty} \qquad \text{and} \qquad d\underline{\psi} = d\alpha d\beta d\lambda d\theta. \tag{33}$$

Then,

$$\pi\left(\underline{\psi} \mid \underline{x}\right) = \mathrm{A}\alpha^{a_1} \beta^{a_1+a_2} \lambda^{a_3} \theta^{a_3+a_4} \left[\prod_{i=1}^{r} h\left(x_{(i)}; \underline{\psi}\right)\right] \left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)}$$
$$\times \exp\left\{-\beta\left(\frac{\alpha}{b_1} + \frac{1}{b_2}\right) - \theta\left(\frac{\lambda}{b_3} + \frac{1}{b_4}\right) - (n-r)\, u_{(r)} - \sum_{i=1}^{r} u_{(i)}\right\}, \tag{34}$$

where
$h\left(x_{(i)}; \underline{\psi}\right)$ is given by (18), $u_{(r)}$ and $w_{(r)}$ are defined in (9) and (10) and $u_{(i)}$ and $w_{(i)}$ are defined in (19) and (20).

The *Bayesian predictors* (BPs) of $Y_{(s)}$ can be derived from the *Bayesian predictive density* (BPD), using the conditional pdf of $Y_{(s)}$ given $\underline{x} = x_{(1)}, \ldots, x_{(r)}$, $f\left(y_{(s)} \mid \underline{x}; \underline{\psi}\right)$, and the joint posterior distribution of $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$, the BPD of $Y_{(s)}$ given $\underline{x}$ can be obtained as:

$$g\left(y_{(s)} \mid \underline{x}\right)=\int_{\underline{\psi}} f\left(y_{(s)} \mid \underline{x};\underline{\psi}\right) \pi\left(\underline{\psi} \mid \underline{x}\right) d\underline{\psi}. \tag{35}$$

Substituting (8) and (34) into (35), then the BPD can be obtained as:

$$\begin{aligned} g\left(y_{(s)} \mid \underline{x}\right)= & \int_{\underline{\psi}} \mathrm{A}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\sum_{j=0}^{s-r-1} C_{r,s,n,j} h\left(y_{(s)};\underline{\psi}\right) \\ & \times \exp\left\{-(n-s+j+1)\left[e^{\alpha y_{(s)}-\frac{\beta}{y_{(s)}}}-u_{(r)}+\theta\ln\left(1+\frac{y_{(s)}}{\lambda}\right)\right.\right. \\ & \left.-\theta\ln w_{(r)}\ \right]-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,u_{(r)} \\ & \left.-\sum_{i=1}^{r} u_{(i)}\right\}\left[\prod_{i=1}^{r} h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)} d\underline{\psi}, \end{aligned} \tag{36}$$

where
$\int_{\underline{\psi}}$ and $d\underline{\psi}$ are given in (33), A is the normalizing constant defined in (32) and $C_{r,s,n,j}$, $h\left(y_{(s)};\underline{\psi}\right)$ and $h\left(x_{(i)};\underline{\psi}\right)$ are given, respectively, by (7), (9) and (18).

a. Point prediction

The BP of $Y_{(s)}$ under the SE loss function, denoted by $\tilde{y}_{(s)SE}$, can be derived as follows:

$$\tilde{y}_{(s)SE}=E\left(y_{(s)} \mid \underline{x}\right)=\int_{y_{(s)}} y_{(s)} g\left(y_{(s)} \mid \underline{x}\right) dy_{(s)}. \tag{37}$$

Using the BPD in (36) and substituting it into (37) as given below

$$\begin{aligned} \tilde{y}_{(s)SE}= & \int_{x_{(r)}}^{\infty}\int_{\underline{\psi}} \mathrm{A}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4} y_{(s)}\sum_{j=0}^{s-r-1} C_{r,s,n,j} h\left(y_{(s)};\underline{\psi}\right) \\ & \times \exp\left\{-(n-s+j+1)\left[e^{\alpha y_{(s)}-\frac{\beta}{y_{(s)}}}-u_{(r)}+\theta\ln\left(1+\frac{y_{(s)}}{\lambda}\right)\right.\right. \\ & \left.\left.-\ \theta\ln w_{(r)}\ \right]-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,u_{(r)}-\sum_{i=1}^{r} u_{(i)}\right\} \\ & \times\left[\prod_{i=1}^{r} h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)} d\underline{\psi}\, dy_{(s)}. \end{aligned} \tag{38}$$

The BP of $Y_{(s)}$ under the LINEX loss function can be derived as follows:

$$\tilde{y}_{(s)LIN}=\frac{-1}{\nu}\ln\left[E\left(e^{-\nu y_{(s)}}\mid \underline{x}\right)\right] , \tag{39}$$

where

$$\begin{aligned}E\left(e^{-\nu y_{(s)}}\mid \underline{x}\right)=&\int_{y_{(s)}} e^{-\nu y_{(s)}} g\left(y_{(s)}\mid \underline{x}\right)dy_{(s)}\\ &=\int_{x_{(r)}}^{\infty}\int_{\underline{\psi}} \mathrm{A}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}e^{-\nu y_{(s)}}\int_{j=0}^{s-r-1} C_{r,s,n,j}h\left(y_{(s)};\underline{\psi}\right)\\ &\times\exp\left\{-\left(n-s+j+1\right)\left[e^{\alpha y_{(s)}-\frac{\beta}{y_{(s)}}}-u_{(r)}+\theta\ln\left(1+\frac{y_{(s)}}{\lambda}\right)\right.\right.\\ &\left.\left.-\ \theta\ln w_{(r)}\ \right]-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-\left(n-r\right)u_{(r)}-\sum_{i=1}^{r}u_{(i)}\right\}\\ &\times\left[\prod_{i=1}^{r}h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r}w_{(i)}^{-\theta}\right]w_{(r)}^{-\theta(n-r)}d\underline{\psi}dy_{(s)}.\end{aligned} \tag{40}$$

Substituting (40) into (39), then the BP under the LINEX loss function can be obtained.

b. Interval prediction

The $(1-\omega)\,100\%$ *Bayesian predictive bounds* (BPBs), $\left(L_{B1}\left(\underline{x}\right),U_{B1}\left(\underline{x}\right)\right)$ of the future order statistic, $Y_{(s)}$, can be obtained using the following probabilities:

$$P\left[Y_{(s)}>L_{B1}\left(\underline{x}\right)|\underline{x}\right]=\int_{L_{B1}(\underline{x})}^{\infty} g\left(y_{(s)}\mid \underline{x}\right)dy_{(s)}=1-\frac{\omega}{2}, \tag{41}$$

and

$$P\left[Y_{(s)}>U_{B1}\left(\underline{x}\right)|\underline{x}\right]=\int_{U_{B1}(\underline{x})}^{\infty} g\left(y_{(s)}\mid \underline{x}\right)dy_{(s)}=\frac{\omega}{2}. \tag{42}$$

Substituting the BPD given by (36) into (41) and (42) and solving numerically for the BPBs.

Remarks

- If $s=r+1$, the conditional predictor, $\hat{y}_{(1)}$, and BP under the SE and LINEX loss functions, $\tilde{y}_{(1)SE}$ and $\tilde{y}_{(1)LIN}$, of the first observation in the future sample can be obtained.
- If $s=\frac{n-r+1}{2}$ (when the future sample size is odd), the conditional predictor, $\hat{y}_{\left(\frac{n-r+1}{2}\right)}$, and BP under the SE and LINEX loss functions, $\tilde{y}_{\left(\frac{n-r+1}{2}\right)SE}$ and $\tilde{y}_{\left(\frac{n-r+1}{2}\right)LIN}$, of the median of the future sample can be obtained.
- If $s=n-r$, the conditional predictor, $\hat{y}_{(n-r)}$, and BP under the SE and LINEX loss functions, $\tilde{y}_{(n-r)SE}$ and $\tilde{y}_{(n-r)LIN}$, of the last observation in the future sample can be obtained.

3. Two-Sample Prediction

This section is devoted to investigate non-Bayesian and Bayesian two-sample prediction (point and interval) for future observation from the AFWE-L distribution based on Type II censoring scheme.

Suppose that $X_{(1)}, X_{(2)}, \cdots, X_{(r)}$ is the observed ordered lifetimes of a censored Type II sample of size n (the informative sample) from the AFWE-L distribution and $Z_{(1)}, Z_{(2)}, \cdots, Z_{(m)}$ is the order statistics of a future sample of size m from AFWE-L distribution. Assuming that the two samples are independent. This section aims to derive different predictors for the future order statistic $Z_{(s)}$, for $1 \leq s \leq m$. The pdf of the s^{th} order statistic from the future sample is defined by:

$$f\left(z_{(s)} \mid \underline{\psi}\right) = C_{s,m} f\left(z_{(s)};\underline{\psi}\right)\left[1 - R\left(z_{(s)};\underline{\psi}\right)\right]^{s-1}\left[R\left(z_{(s)};\underline{\psi}\right)\right]^{m-s}, \qquad z_{(s)} > 0, \qquad (43)$$

were

$$C_{s,m} = \frac{m!}{(s-1)!\,(m-s)!}.$$

Since

$$f\left(z_{(s)};\underline{\psi}\right) = h\left(z_{(s)};\underline{\psi}\right) R\left(z_{(s)};\underline{\psi}\right),$$

Using the binomial expansion of $\left[1 - R\left(z_{(s)};\underline{\psi}\right)\right]^{s-1}$.
Hence $\left[1 - R\left(z_{(s)};\underline{\psi}\right)\right]^{s-1} = \sum_{j=0}^{s-1}\binom{s-1}{j}(-1)^j\left[R\left(z_{(s)};\underline{\psi}\right)\right]^j$.
Then (43) can be rewritten as follows:

$$\left(z_{(s)} \mid \underline{\psi}\right) = \sum_{j=0}^{s-1} C_{s,m,j} h\left(z_{(s)};\underline{\psi}\right)\left[R\left(z_{(s)};\underline{\psi}\right)\right]^{j+m-s+1}, \qquad z_{(s)} > 0, \qquad (44)$$

where

$$C_{s,m,j} = \frac{m!(-1)^j}{j!\;(s-j-1)!\,(m-s)!} \qquad (45)$$

Substituting (3) and (4) into (44), then the pdf of the s^{th} order statistic from the future sample is given by:

$$\begin{aligned} f\left(z_{(s)} \mid \underline{\psi}\right) = \sum_{j=0}^{s-1} C_{s,m,j} h\left(z_{(s)};\underline{\psi}\right) \exp\Bigg\{-(j+m-s+1)\Bigg[e^{\alpha z_{(s)} - \frac{\beta}{z_{(s)}}} \\ +\theta \ln\left(1+\frac{z_{(s)}}{\lambda}\right)\Bigg]\Bigg\}, \qquad z_{(s)} > 0, \end{aligned} \qquad (46)$$

where
$C_{s,m,j}$ is defined in (45) and

$$h\left(z_{(s)};\underline{\psi}\right)=\left[\left(\alpha+\frac{\beta}{z_{(s)}^{2}}\right)e^{\alpha z_{(s)}-\frac{\beta}{z_{(s)}}}+\frac{\theta}{\lambda}\left(1+\frac{z_{(s)}}{\lambda}\right)^{-1}\right]. \tag{47}$$

3.1. Conditional prediction approach

In this subsection, two-sample conditional prediction approach is used to present point and interval predictor of $Z_{(s)}$.

a. Point prediction

The conditional predictor of $Z_{(s)}$; $\hat{z}_{(s)}$, can be derived using the pdf of the s^{th} order statistic of the future sample in (46). Then using the invariance property of the ML estimators, the parameters $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$ will be replaced by their ML estimators, $\underline{\widehat{\psi}}=\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda},\widehat{\theta}\right)$, which were derived in Salem *et al.* (2022) as follows:

$$f\left(z_{(s)}\,\middle|\,\underline{\widehat{\psi}}\right)=\sum_{j=0}^{s-1}C_{s,m,j}h\left(z_{(s)};\underline{\widehat{\psi}}\right),$$
$$\times\exp\left\{-\left(j+m-s+1\right)\left[e^{\widehat{\alpha}z_{(s)}-\frac{\widehat{\beta}}{z_{(s)}}}+\widehat{\theta}\ln\left(1+\frac{z_{(s)}}{\widehat{\lambda}}\right)\right]\right\},\ z_{(s)}>0. \tag{48}$$

where
$C_{s,m,j}$ is given by (45) and

$$h\left(z_{(s)};\underline{\widehat{\psi}}\right)=\left(\widehat{\alpha}+\frac{\widehat{\beta}}{z_{(s)}^{2}}\right)e^{\widehat{\alpha}z_{(s)}-\frac{\widehat{\beta}}{z_{(s)}}}+\frac{\widehat{\theta}}{\widehat{\lambda}}\left(1+\frac{z_{(s)}}{\widehat{\lambda}}\right)^{-1}. \tag{49}$$

Then, the conditional predictor of $Z_{(s)}$ can be obtained as follows:

$$\hat{z}_{(s)}=E\left(Z_{(s)}\,\middle|\,\underline{\widehat{\psi}}\right)=\int_{z_{(s)}}z_{(s)}f\left(z_{(s)}\,\middle|\,\underline{\widehat{\psi}}\right)dz_{(s)}$$
$$=\int_{0}^{\infty}z_{(s)}\sum_{j=0}^{s-1}C_{s,m,j}h\left(z_{(s)};\underline{\widehat{\psi}}\right)$$
$$\times\exp\left\{-\left(j+m-s+1\right)\left[e^{\widehat{\alpha}z_{(s)}-\frac{\widehat{\beta}}{z_{(s)}}}+\widehat{\theta}\ln\left(1+\frac{z_{(s)}}{\widehat{\lambda}}\right)\right]\right\}dz_{(s)}\,. \tag{50}$$

Numerical method can be used to obtain $\hat{z}_{(s)}$.

b. Interval prediction

Using the pdf of the s^{th} order statistic given $\underline{x}=x_{(1)},\ ...,\ x_{(r)}$ and by replacing the parameters $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$ by their ML estimators $\underline{\widehat{\psi}}=\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda},\widehat{\theta}\right)$ as in (48), the

$(1-\omega)\,100\%$ conditional predictive interval for the future ordered failure, $Z_{(s)}$ can be derived from the following probability:

$$P\left[L_{C2}\left(\underline{x}\right) < Z_{(s)} < U_{C2}\left(\underline{x}\right)|\underline{x}\right] = 1-\omega, \tag{51}$$

where $L_{C2}\left(\underline{x}\right)$ and $U_{C2}\left(\underline{x}\right)$ are the lower and upper bounds of the prediction interval based on the conditional prediction approach.
The conditional predictive bounds, $\left(L_{C2}\left(\underline{x}\right),\ U_{C2}\left(\underline{x}\right)\right)$, can be obtained as given below

$$P\left[Z_{(s)} > L_{C2}\left(\underline{x}\right)|\underline{x}\right] = \int_{L_{C2}(\underline{x})}^{\infty} f\left(z_{(s)}\;\middle|\;\widehat{\underline{\psi}}\right) dz_{(s)} = 1-\frac{\omega}{2}, \tag{52}$$

and

$$P\left[Z_{(s)} > U_{C2}\left(\underline{x}\right)|\underline{x}\right] = \int_{U_{C2}(\underline{x})}^{\infty} f\left(z_{(s)}\;\middle|\;\widehat{\underline{\psi}}\right) dz_{(s)} = \frac{\omega}{2}. \tag{53}$$

Substituting (48) into (52) and (53), then the lower and upper bounds of the prediction interval of $Z_{(s)}$ can be obtained by solving numerically.

3.2. Bayesian prediction

In this subsection, two-sample Bayesian prediction of a future order statistic, $Z_{(s)}$, from AFWE-L distribution based on Type II censoring scheme is considered under the SE loss function as a symmetric loss function and under the LINEX loss function as an asymmetric loss function. Moreover, credible interval of $Z_{(s)}$ is obtained.
The BPs of $Z_{(s)}$ can be derived from the BPD of $Z_{(s)}$. Using the pdf of $Z_{(s)}$, and the joint posterior distribution of $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$; $\pi\left(\underline{\psi}\mid\underline{x}\right)$, the BPD of $Z_{(s)}$ given $\underline{x}$ is defined by:

$$g\left(z_{(s)}\mid\underline{x}\right) = \int_{\underline{\psi}} f\left(z_{(s)}\mid\underline{\psi}\right)\pi\left(\underline{\psi}\mid\underline{x}\right) d\underline{\psi}. \tag{54}$$

Substituting (46) and (34) into (54), then the BPD can be obtained as follows:

$$\begin{aligned} g\left(z_{(s)}\mid\underline{x}\right) = \int_{\underline{\psi}} & \mathrm{A}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4}\sum_{j=0}^{s-1} C_{s,m,j}\,h\left(z_{(s)};\underline{\psi}\right) \\ & \times\exp\left\{-\left(j+m-s+1\right)\left[e^{\alpha z_{(s)}-\frac{\beta}{z_{(s)}}}+\theta\ln\left(1+\frac{z_{(s)}}{\lambda}\right)\right]\right\} \\ & \times\left[\prod_{i=1}^{r} h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)}\exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)\right. \\ & \left.-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-\left(n-r\right)u_{(r)}-\sum_{i=1}^{r} u_{(i)}\right\}\, d\underline{\psi}, \end{aligned} \tag{55}$$

where
$\int_{\underline{\psi}}$ and $d\underline{\psi}$ are given in (33), A is the normalizing constant defined in (32), and $C_{s,m,j}$, $h\left(z_{(s)};\underline{\psi}\right)$ and $h\left(x_{(i)};\underline{\psi}\right)$ are defined in (45), (47) and (18) and $u_{(r)}$, $w_{(r)}$, $u_{(i)}$ and $w_{(i)}$ are given, respectively, in (9), (10), (19) and (20).

a. Point prediction

The BP of $Z_{(s)}$ under the SE loss function can be derived as follows:

$$\begin{aligned}\tilde{z}_{(s)SE} = E\left(z_{(s)} \mid \underline{x}\right) &= \int_{z_{(s)}} z_{(s)} g\left(z_{(s)} \mid \underline{x}\right) dz_{(s)} \\ &= \int_0^\infty \int_{\underline{\psi}} \mathrm{A}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4} z_{(s)} \sum_{j=0}^{s-1} C_{s,m,j} h\left(z_{(s)};\underline{\psi}\right) \\ &\times\exp\left\{-\left(j+m-s+1\right)\left[e^{\alpha z_{(s)}-\frac{\beta}{z_{(s)}}}+\theta\ln\left(1+\frac{z_{(s)}}{\lambda}\right)\right]\right\} \\ &\times\left[\prod_{i=1}^{r} h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)}\exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)\right. \\ &\left.-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,u_{(r)}-\sum_{i=1}^{r} u_{(i)}\right\} d\underline{\psi}dz_{(s)}.\end{aligned} \tag{56}$$

By solving (56) numerically, the BP of $Z_{(s)}$ under the SE loss function can be obtained. The BP of $Z_{(s)}$ under the LINEX loss function can be derived as given below:

$$\tilde{z}_{(s)LIN} = \frac{-1}{\nu}\ln\left[E\left(e^{-\nu z_{(s)}} \mid \underline{x}\right)\right], \tag{57}$$

where

$$\begin{aligned}E\left(e^{-\nu z_{(s)}} \mid \underline{x}\right) &= \int_{z_{(s)}} e^{-\nu z_{(s)}} g\left(z_{(s)} \mid \underline{x}\right) dz_{(s)} \\ &= \int_0^\infty \int_{\underline{\psi}} \mathrm{A}\alpha^{a_1}\beta^{a_1+a_2}\lambda^{a_3}\theta^{a_3+a_4} z_{(s)} \sum_{j=0}^{s-1} C_{s,m,j} h\left(z_{(s)};\underline{\psi}\right) \\ &\times\exp\left\{-\left(j+m-s+1\right)\left[e^{\alpha z_{(s)}-\frac{\beta}{z_{(s)}}}+\theta\ln\left(1+\frac{z_{(s)}}{\lambda}\right)\right]\right\} \\ &\times\left[\prod_{i=1}^{r} h\left(x_{(i)};\underline{\psi}\right)\right]\left[\prod_{i=1}^{r} w_{(i)}^{-\theta}\right] w_{(r)}^{-\theta(n-r)}\exp\left\{-\beta\left(\frac{\alpha}{b_1}+\frac{1}{b_2}\right)\right. \\ &\left.-\theta\left(\frac{\lambda}{b_3}+\frac{1}{b_4}\right)-(n-r)\,u_{(r)}-\sum_{i=1}^{r} u_{(i)}\right\} d\underline{\psi}dz_{(s)}.\end{aligned} \tag{58}$$

Substituting (58) into (57), then the BP under the LINEX loss function can be obtained.

b. Interval prediction

The $(1-\omega)\,100\%$ BPBs, $(L_{B2}(\underline{x}), U_{B2}(\underline{x}))$ of the future order statistic, $Z_{(s)}$, can be obtained using the following probabilities:

$$P\left[Z_{(s)} > L_{B2}(\underline{x})\,|\underline{x}\right] = \int_{L_{B2}(\underline{x})}^{\infty} g\left(z_{(s)} \mid \underline{x}\right) dz_{(s)} = 1 - \frac{\omega}{2}, \tag{59}$$

and

$$P\left[Z_{(s)} > U_{B2}(\underline{x})\,|\underline{x}\right] = \int_{U_{B2}(\underline{x})}^{\infty} g\left(z_{(s)} \mid \underline{x}\right) dz_{(s)} = \frac{\omega}{2}. \tag{60}$$

Substituting the BPD in (55) into (59) and (60) the BPBs of $Z_{(s)}$ can be obtained.

Remarks

- If $s = 1$, the conditional predictor, $z_{(1)}$, and BP under the SE and LINEX loss functions, $\tilde{z}_{(1)SE}$ and $\tilde{z}_{(1)LIN}$, of the first observation in the future sample can be obtained.
- If $s = \frac{m+1}{2}$ (when the future sample size is odd), the conditional predictor, $\hat{z}_{\left(\frac{m+1}{2}\right)}$, and BP under the SE and LINEX loss functions, $\tilde{z}_{\left(\frac{m+1}{2}\right)SE}$ and $\tilde{z}_{\left(\frac{m+1}{2}\right)LIN}$, of the median of the future sample can be obtained.
- If $s = m$, the conditional predictor, $\hat{z}_{(m)}$, and BP under the SE and LINEX loss functions, $\tilde{z}_{(m)SE}$ and $\tilde{z}_{(m)LIN}$, of the last observation in the future sample can be obtained.

4. Simulation Study

In this section, a simulation study is conducted to evaluate the performance of the different predictors of future observation based on one-sample and two-sample prediction schemes. Also, the performance of BPs under the SE loss function is compared with the BPs under the LINEX loss function based on the conducted simulation study.

4.1. Conditional prediction approach

- For evaluating the performance of the conditional predictors of the future observation from the AFWE-L distribution under 30% level of Type II censoring scheme based on one-sample and two-sample prediction the following steps are used:

a Generate random samples of size $n = 30$ from the AFWE-L distribution using three combinations of the population parameter values:

$$\text{I} : (\alpha = 1.15, \beta = 0.3, \lambda = 0.15, \theta = 0.1)\,,$$

$$\text{II}: \ (\alpha{=}\ 0.8,\beta{=}\ 0.5,\lambda{=}\ 0.5,\theta{=}\ 0.5)\,,$$

and

$$\text{III}: (\alpha = 0.5, \beta = 0.25, \lambda = 0.15, \theta = 0.1)\,.$$

b. Substituting the ML estimates of the parameters, $\underline{\widehat{\psi}} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\theta}\right)$, into the equation of the conditional pdf of the s^{th} order statistic given $\underline{x} = x_{(1)}, \ldots, x_{(r)}$ in the case of one-sample prediction and for given values of s, where $r < s \leq n$, the conditional prediction for the future observation $\hat{y}_{(s)C}$ can be computed based on 30% level of Type-II censoring.
c. Substituting the ML estimates of the parameters, $\underline{\widehat{\psi}} = \left(\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\theta}\right)$, into the equation of the pdf of the s^{th} order statistic in the case of two-sample prediction and for given values of m (future sample size) and s, where $1 \leq s \leq m$, the conditional prediction for the future observation $\hat{z}_{(s)C}$ can be computed based on 30% level of Type-II censoring.
d. The simulation study is conducted using *number of replications* (NR), $NR = 1000$ using Mathematica 11.

Tables 1 - 3 display the conditional predictors of the future observation from the AFWE-L distribution and the bounds of the conditional intervals of the future observation along with their lengths based on the one-sample prediction scheme.

Tables 4 - 6 present the conditional predictors of the future observation from the AFWE-L distribution and the bounds of the conditional intervals of the future observation along with their lengths based on the two-sample prediction scheme.

Table 1

One-sample conditional predictors and 95% conditional intervals of future observation from AFWE-L distribution along with their lengths for different sample size $n = 30$, $r = 21$, $NR = 1000$ and $(\alpha = 1.15, \beta = 0.3, \lambda = 0.15, \theta = 0.1)$

s	$\widehat{y}_{(s)c}$	LL	UL	Length
22	0.6476	0.5341	0.7151	0.1810
26	1.1505	0.9299	1.2851	0.3552
30	1.6492	1.0585	1.9016	0.8432

Table 2

One-sample conditional predictors and 95% conditional intervals of future observation from AFWE-L distribution along with their lengths for sample size $n = 30$, $r = 21$, $NR = 1000$ and $(\alpha = 0.8, \beta = 0.5, \lambda = 0.5, \theta = 0.5)$

s	$\widehat{y}_{(s)c}$	LL	UL	Length
22	0.2472	0.1375	0.3674	0.2299
26	0.9137	0.5612	1.1137	0.5525
30	2.1604	1.4445	2.3159	0.8714

Table 3

One-sample conditional predictors and 95% conditional intervals of future observation from AFWE-L distribution along with their lengths for sample size $n = 30$, $r = 21$, $NR = 1000$ and $(\alpha = 0.5, \beta = 0.25, \lambda = 0.15, \theta = 0.1)$

s	$\widehat{y}_{(s)c}$	**LL**	**UL**	**Length**
22	1.2792	0.9426	1.4170	0.4744
26	2.6866	2.0634	2.9511	0.8877
30	4.3581	3.1065	4.8742	1.7677

Table 4

Two-sample conditional predictors and 95% conditional intervals of future observation from AFWE-L distribution along with their lengths for sample size $n = 30$ and $m = 15$, $r = 21$, $NR = 1000$ and $(\alpha = 1.15, \beta = 0.3, \lambda = 0.15, \theta = 0.1)$

s	$\widehat{z}_{(s)c}$	**LL**	**UL**	**Length**
1	0.2115	0.1032	0.2454	0.1422
8	0.6282	0.3847	0.7603	0.3756
15	1.8947	1.1846	1.9386	0.7541

Table 5

Two-sample conditional predictors and 95% conditional intervals of future observation from AFWE-L distribution along with their lengths for sample size $n = 30$ and $m = 15$, $r = 21$, $NR = 1000$ and $(\alpha = 0.8, \beta = 0.5, \lambda = 0.5, \theta = 0.5)$

s	$\widehat{z}_{(s)c}$	**LL**	**UL**	**Length**
1	0.2027	0.1055	0.2778	0.1723
8	0.7647	0.4552	1.0412	0.5860
15	2.6858	1.6797	2.8802	1.2005

Table 6

Two-sample conditional predictors and 95% conditional intervals of future observation from AFWE-L distribution along with their lengths for sample size $n = 30$ and $m = 15$, $r = 21$, $NR = 1000$ and $(\alpha = 0.5, \beta = 0.25, \lambda = 0.15, \theta = 0.1)$

s	$\widehat{z}_{(s)c}$	**LL**	**UL**	**Length**
1	0.1675	0.0839	0.2259	0.1420
8	0.7741	0.4142	1.2081	0.7939
15	3.7036	2.2493	4.0421	1.7928

4.2. Bayesian prediction

For evaluating the performance of the Bayes predictors of future observation from the AFWE-L distribution under 30% level of Type II censoring scheme based on one-sample and two-sample prediction, the following steps are used:

b To generate several random samples from the AFWE-L distribution, the following steps are used:
. Three combinations of the population parameter value $\underset{\sim}{\psi} = (\alpha, \beta, \lambda, \theta)$, are considered as follows:

$$I : (\alpha = 0.8, \quad \beta = 1.15, \quad \lambda = 1.5, \quad \theta = 2)\,,$$

$$II : (\alpha = 0.5, \quad \beta = 0.3, \quad \lambda = 0.5, \quad \theta = 0.3) ,$$

and

$$III : (\alpha = 1.15, \quad \beta = 0.3, \quad \lambda = 0.15, \quad \theta = 0.8) ,$$

and sample size $n = 30$.
. Generate n independent random variables X_{1k} from the FWE distribution (α, β).
.Generate n independent random variables X_{2k} from L distribution with (λ, θ).
. From Steps 1, 2 and 3, n independent random variables X_k from AFWE-L distribution can be obtained as:

$$X_k = \min (X_{1k}, X_{2k}) \ ,$$

where X_{1k} and X_{2k} are independent of each other.

c For given values of the hyperparameters, generate α and λ from the conditional gamma prior distributions and generate β and θ from gamma prior distribution.

d For the one-sample prediction the conditional pdf of the s^{th} order statistic given$\underline{x} = x_{(1)}, \ . \ . \ . \ , \ x_{(r)}$ for given values of s, where $r < s \leq n$, is used for evaluating the BPD of a future observation, $Y_{(s)}$, from the AFWE-L distribution.

e For the two-sample prediction the pdf of the s^{th} order statistic for given values of s and the future sample size m, where $1 \leq s \leq m$, is used for evaluating the BPD of a future observation, $Z_{(s)}$, from the AFWE-L distribution.

f Based on Steps c and d, the BPs are calculated based on the SE and LINEX loss functions. Also, the BPBs along with their lengths are evaluated.

g The simulation study is conducted using $NR = 10000$ using R programming language.

Tables 7 - 9 display the BPs and BPBs of future observations from the AFWE-L distribution and along with their lengths based on the one-sample prediction scheme. Whereas Tables 10 - 12 present the BPs and BPBs of future observations from the AFWE-L distribution along with their lengths based on the two-sample prediction scheme.

Table 7

One-sample Bayes predictors and 95% Bayesian prediction bounds of future observations from AFWE-L distribution along with their lengths under SE and LINEX loss functions for sample size $n = 30$, $r = 21$, $NR = 10000$

s	SE loss function				LINEX loss function			
	$\widetilde{y}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{y}_{(s)LIN}$	**LL**	**UL**	**Length**
22	0.4875	0.4763	0.4980	0.0217	0.4986	0.4911	0.5077	0.0166
26	0.7849	0.7763	0.7998	0.0235	0.7947	0.7853	0.8023	0.0169
30	1.1938	1.1822	1.2057	0.0235	1.1969	1.1850	1.2031	0.0181

Table 8

One-sample Bayes predictors and 95% Bayesian prediction bounds of future observations from AFWE-L distribution along with their lengths under SE and LINEX loss functions for sample size $n = 30$, $r = 21$, $NR = 10000$

s	SE loss function				LINEX loss function			
	$\widetilde{y}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{y}_{(s)LIN}$	**LL**	**UL**	**Length**
22	0.4875	0.4763	0.4980	0.0217	0.4986	0.4911	0.5077	0.0166
26	0.7849	0.7763	0.7998	0.0235	0.7947	0.7853	0.8023	0.0169
30	1.1938	1.1822	1.2057	0.0235	1.1969	1.1850	1.2031	0.0181

Table 9

One-sample Bayes predictors and 95% Bayesian prediction bounds of future observations from AFWE-L distribution along with their lengths under SE and LINEX loss functions for sample size $n = 30$, $r = 21$, $NR = 10000$

s	SE loss function				LINEX loss function			
	$\widetilde{y}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{y}_{(s)LIN}$	**LL**	**UL**	**Length**
22	1.4938	1.4823	1.5031	0.0208	1.4948	1.4833	1.5033	0.0200
26	2.0859	2.0753	2.0993	0.0240	2.0917	2.0820	2.1035	0.0215
30	3.2337	3.2193	3.2490	0.0297	3.2407	3.2286	3.2552	0.0265

Table 10

Two-sample Bayes predictors and 95% Bayesian prediction bounds of future observations from AFWE-L distribution along with their lengths under SE and LINEX loss functions for sample size $n = 30$ and $m = 15$, $r = 21$, $NR = 10000$

s	SE loss function				LINEX loss function			
	$\widetilde{z}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{z}_{(s)LIN}$	**LL**	**UL**	**Length**
1	0.2918	0.2846	0.3023	0.0178	0.2976	0.2922	0.3039	0.0117
8	0.5882	0.5741	0.5985	0.0244	0.5982	0.5899	0.6049	0.0150
15	0.9167	0.9009	0.9270	0.0261	0.8964	0.8859	0.9034	0.0174

Table 11

Two-sample Bayes predictors and 95% Bayesian prediction bounds of future observations from AFWE-L distribution along with their lengths under SE and LINEX loss functions for sample size $n = 30$ and $m = 15$, $r = 21$, $NR = 10000$

s	SE loss function				LINEX loss function			
	$\widetilde{z}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{z}_{(s)LIN}$	**LL**	**UL**	**Length**
1	1.2030	1.1933	1.2197	0.0264	1.1977	1.1831	1.2077	0.0246
8	1.5050	1.4894	1.5184	0.0290	1.4912	1.4781	1.5037	0.0256
15	1.9920	1.9804	2.0108	0.0304	1.9926	1.9780	2.0052	0.0272

Table 12
Two-sample Bayes predictors and 95% Bayesian prediction bounds of future observations from AFWE-L distribution with their lengths under SE and LINEX loss functions for sample size $n = 30$ and $m = 15$, $r = 21$, $NR = 10000$

s	SE loss function				LINEX loss function			
	$\widetilde{z}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{z}_{(s)LIN}$	**LL**	**UL**	**Length**
1	0.1072	0.0978	0.1141	0.0163	0.0949	0.0880	0.1024	0.0144
8	0.2807	0.2679	0.2932	0.0253	0.2968	0.2884	0.3055	0.0171
15	0.7900	0.7752	0.8042	0.0290	0.7936	0.7833	0.8072	0.0240

5. Applications

This section is devoted to demonstrating the applicability and flexibility of the AFWE-L distribution for data modeling and how the different presented predictors can be used in practice through three applications of COVID-19 data in some countries that were used in Salem *et al.*(2022).

Application 1

This application is presented by Mubarak and Almetwally (2021). The data represent a COVID-19 data belong to the United Kingdom of 76 days, from 15 April to 30 June 2020. these data formed of drought mortality rates. The data are: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019 and 11.4584.

The AFWE-L distribution can be used for modelling this data set, since the Kolmogorov Smirnov ($K{-}S$) test statistic and its corresponding p-value were 0.0790 and 0.9735.

In this application, the one-sample prediction scheme is used for predicting the remaining observations, $(n - r)$, under 30% level of Type II censoring. Tables 13 and 14 present the one-sample conditional and Bayes predictors of $Y_{(s)}$, where $r < s \leq n$. Also, the prediction bounds and their lengths are obtained.
Moreover, the two-sample prediction is used for predicting a future observation, $Z_{(s)}$, from a future sample with size m of drought mortality rates of COVID-19 in the United Kingdom, where $1 \leq s \leq m$. Tables 15 and 16 display the two-sample conditional and Bayes predictor of $Z_{(s)}$, where $1 \leq s \leq m$. Also, the prediction bounds and their lengths are presented.

Table 13
One-Sample conditional prediction and 95% condition interval bounds and their lengths of the future observation for COVID-19 data of the United Kingdom

n	r	s	$\widehat{y}_{(s)c}$	**LL**	**UL**	**Length**
76	53	54	3.3146	2.4705	3.3277	0.8573
		65	7.9910	5.4855	8.7079	2.2224
		76	20.7681	14.1763	21.5442	7.3679

Table 14
One-Sample Bayes predictors and 95% Bayesian prediction bounds with their lengths of the future observation for COVID-19 data of the United Kingdom

n	r	s	SE loss function				LINEX loss function			
			$\widetilde{y}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{y}_{(s)LIN}$	**LL**	**UL**	**Length**
76	53	54	3.3638	3.3526	3.3752	0.0226	3.3677	3.3584	3.3801	0.0217
		65	7.0809	7.0636	7.0948	0.0312	7.0611	7.0501	7.0763	0.0263
		76	11.4780	11.4591	11.4909	0.0318	11.4497	11.4385	11.4660	0.0275

Table 15
Two-Sample conditional predictors and 95% conditional interval bounds with their lengths of the future observation for COVID-19 data of the United Kingdom

n, m	r	s	$\widehat{z}_{(s)c}$	**LL**	**UL**	**Length**
76, 35	53	1	0.2684	0.1336	0.2797	0.1462
		18	2.0806	1.0922	3.1727	2.0805
		35	21.0310	13.7513	21.1314	7.3801

Table 16
Two-Sample Bayes predictors and 95% Bayesian prediction bounds with their lengths of the future observation for COVID-19 data of the United Kingdom

n, m	r	s	SE loss function				LINEX loss function			
			$\widetilde{z}_{(s)SE}$	**LL**	**UL**	**Length**	$\widetilde{z}_{(s)LIN}$	**LL**	**UL**	**Length**
76, 35	53	1	0.4930	0.4831	0.5022	0.0191	0.4970	0.4883	0.5058	0.0175
		18	1.5091	1.4983	1.5194	0.0212	1.4999	1.4905	1.5101	0.0196
		35	4.0058	3.9955	4.0188	0.0234	4.0089	3.9986	4.0206	0.0221

Application 2

This application is given by Mubarak and Almetwally (2021). The data represent a COVID-19 data belong to Japan of 38 days, from 4 September to 12 October 2020. These data formed of drought mortality rates. The data are: 0.1596, 0.2733, 0.1142, 0.0851, 0.1976, 0.2243, 0.1810, 0.0828, 0.1504, 0.2169, 0.0404, 0.1208, 0.1334, 0.1589, 0.1184, 0.1698, 0.0648, 0.1027, 0.0511, 0.1019, 0.1520, 0.1006, 0.0624, 0.0372, 0.1112, 0.0859, 0.0854, 0.0847, 0.1443, 0.0836, 0.0238, 0.0355, 0.0353, 0.0937, 0.0349, 0.0924, 0.0344 and 0.0228.
From the value of $K-S = 0.1316$ and its corresponding $p-value = 0.9033$, then the AFWE-L distribution fits the data very well.

In this application, the one-sample prediction scheme is used for predicting the remaining observations, $(n-r)$ under 30% level of Type II censoring. Tables 17 and 18 present the one-sample conditional and Bayes predictors of $Y_{(s)}$, where $r < s \leq n$. Also, the prediction bounds and their lengths are obtained.

Moreover, the two-sample prediction is used for predicting a future observation, $Z_{(s)}$, from a future sample of COVID-19 drought mortality rates in Japan with size m, where $1 \leq s \leq m$. Tables 19 and 20 display the two-sample conditional and Bayes predictors of $Z_{(s)}$, where $1 \leq s \leq m$. Also, the prediction bounds and their lengths are presented.

Table 17
One-Sample conditional predictors and 95% conditional interval bounds and their lengths of the future observation for COVID-19 data of Japan

n	r	s	$\widehat{y}_{(s)c}$	LL	UL	Length
38	27	28	0.2594	0.1725	0.3221	0.1496
		33	0.3992	0.3240	0.4534	0.1294
		38	0.5591	0.4614	0.6293	0.1679

Table 18
One-Sample Bayes predictors and 95% Bayesian prediction bounds with their lengths of the future observation for COVID-19 data of Japan

n	r	s	SE loss function				LINEX loss function			
			$\widetilde{y}_{(s)SE}$	LL	UL	Length	$\widetilde{y}_{(s)LIN}$	LL	UL	Length
38	27	28	0.1306	0.1220	0.1421	0.0201	0.1367	0.1307	0.1441	0.0134
		33	0.1778	0.1615	0.1847	0.0231	0.1663	0.1564	0.1759	0.0195
		38	0.2607	0.2495	0.2753	0.0257	0.2713	0.2597	0.2840	0.0243

Table 4.19
Two-Sample conditional predictors and 95% conditional interval bounds with their lengths of the future observation for COVID-19 data of Japan

n, m	r	s	$\widehat{z}_{(s)c}$	LL	UL	Length
38,15	27	1	0.2723	0.1655	0.3000	0.1345
		8	0.4078	0.3357	0.4436	0.1079
		15	0.5859	0.4847	0.6365	0.1518

Table 20
Two-Sample Bayes predictors and 95% Bayesian prediction bounds with their lengths of the future observation for COVID-19 data of Japan

n, m	r	s	SE loss function				LINEX loss function			
			$\widetilde{z}_{(s)SE}$	LL	UL	Length	$\widetilde{z}_{(s)LIN}$	LL	UL	Length
38,15	27	1	0.0121	0.0036	0.0241	0.0206	0.0194	0.0100	0.0267	0.0167
		8	0.0444	0.0325	0.0554	0.0229	0.0486	0.0391	0.0597	0.0206
		15	0.9889	0.9770	1.0011	0.0241	0.9968	0.9866	1.0106	0.0240

Application 3

This application is given by Liu *et al.* (2021), where the survival times of patients suffering from the COVID-19 epidemic in China is considered. The data represent the survival times of patients from the time admitted to the hospital until death. Among them, a group of 53 COVID-19 patients were found in critical condition in hospital from January to February 2020.

The data are given by: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

Based on the value of the $K-S = 0.0943$ and its corresponding $p-value = 0.9747$, hence the AFWE-L distribution provides a good fitting to this real data set.

In this application, one-sample prediction scheme is applied under 30% level of

Type II censoring for predicting the remaining observations, $(n-r)$. Tables 21 and 22 present the one-sample conditional and Bayes predictors of $Y_{(s)}$, where $r < s \leq n$. Also, the prediction bounds and their lengths are obtained.

Moreover, the two-sample prediction is used for predicting a future observation, $Z_{(s)}$, from a future sample of COVID-19 drought mortality rates in China with size m, where1 $\leq s \leq m$. Tables 23 and 24 display the two-sample conditional and Bayes predictors of $Z_{(s)}$, where $1 \leq s \leq m$. Also, the prediction bounds and their lengths are presented.

Table 21

One-Sample conditional predictors and 95% conditional interval bounds and their lengths of the future observation for COVID-19 data of China

n	r	s	$\widehat{y}_{(s)c}$	LL	UL	Length
53	37	38	7.6991	5.2161	7.9273	2.7112
		45	17.0940	10.4736	17.9421	7.4685
		46	19.0603	11.6873	19.7235	8.0362
		53	49.0167	29.3250	49.2677	19.9427

Table 22

One-Sample Bayes predictors and 95% Bayesian prediction bounds with their lengths of the future observation for COVID-19 data of China

n	r	s	SE loss function				LINEX loss function			
			$\widetilde{y}_{(s)SE}$	LL	UL	Length	$\widetilde{y}_{(s)LIN}$	LL	UL	Length
53	37	38	5.0701	5.0544	5.0842	0.0298	5.0780	5.0614	5.0874	0.0260
		45	10.8110	10.7932	10.8271	0.0339	10.8157	10.7993	10.8297	0.0304
		46	11.2901	11.2783	11.3142	0.0359	11.2753	11.2552	11.2901	0.0349
		53	20.0934	20.0745	20.1113	0.0368	20.0763	20.0537	20.0890	0.0353

Table 23

Two-Sample conditional predictors and 95% conditional interval bounds with their lengths of the future observation for COVID-19 data of China

n, m	r	s	$\widehat{z}_{(s)c}$	LL	UL	Length
53,25	37	1	5.0659	1.0977	7.0000	5.9023
		13	14.9376	6.2585	29.1000	22.8415
		25	84.7053	32.7065	90.1200	57.4135

Table 24

Two-Sample Bayes predictors and 95% Bayesian prediction bounds with their lengths of the future observation for COVID-19 data of China

n, m	r	s	SE loss function				LINEX loss function			
			$\widetilde{z}_{(s)SE}$	LL	UL	Length	$\widetilde{z}_{(s)LIN}$	LL	UL	Length
53,25	37	1	0.0569	0.0476	0.0657	0.0181	0.0470	0.0383	0.0537	0.0153
		13	0.6071	0.5946	0.6216	0.0270	0.5991	0.5843	0.6066	0.0223
		25	2.5809	2.5723	2.5997	0.0274	2.5993	2.5850	2.6083	0.0233

Concluding remarks

- As s increases, the conditional predictors and the BPs under the SE and LINEX loss functions increase.

- For all prediction intervals, the lower and upper bounds and their lengths increase as s increases, that is the lower and upper bounds and their length of the first order statistic (in the case of two-sample prediction) are less than the lower and upper bounds and their length of the last order statistic, and in the case of one-sample prediction, the lower and upper bounds and their lengths of the $(r+1)^{th}$ order statistic are smaller than their corresponding values for the $(n-r)^{th}$ order statistic.
- The conditional predictions and the BPs under the SE and LINEX loss functions are included between the lower and upper bounds of all prediction intervals.
- In all cases, the lengths of the BPBs under the LINEX loss function are less than the lengths of the BPBs under the SE loss function.

6. Conclusion

Prediction of future observations is an important problem in many practical applications. This paper focuses on deriving different predictors for future observation from the AFWE-L distribution based on Type II censoring scheme. One-sample prediction and two-sample prediction are studied throughout non-Bayesian and Bayesian frameworks. The conditional prediction approach is discussed as a non-Bayesian prediction method. Also, Bayes predictors are obtained by applying a joint bivariate prior distribution under two different loss functions, the SE and LINEX loss functions. Moreover, a simulation study is conducted to evaluate the performance of the derived predictors and three applications of COVID -19 data in some countries are considered. In general, from the simulation study and the applications computations showed that as s increases, all the point predictors increase. Also, for all prediction intervals, the lower and upper bounds and their lengths increase as s increases. The conditional predictors and the BPs under the SE and LINEX loss functions are included between the lower and the upper bounds of all prediction intervals. Finally, in Bayesian approach, the lengths of the BPBs under the LINEX loss function are less than the lengths of the BPBs under the SE loss function.

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